Model for a one-kink metric

1973 J. Phys. A: Math. Nucl. Gen. 6 1

(http://iopscience.iop.org/0301-0015/6/1/002)

J. Phys. A: Math., Nucl. Gen., Vol. 6, January 1973. Printed in Great Britain. © 1973

Model for a one-kink metric

J G Williams† and R K P Zia‡

Department of Mathematical Physics, University of Birmingham, Birmingham, UK

MS received 11 September 1972

Abstract. It is well known that the metric tensor of general relativity may be associated with a conserved quantity, which is an integer known as the 'kink number'. This paper introduces a model for a metric corresponding to a single stationary kink centred at the origin. The Christoffel symbols and the energy–momentum tensor are deduced, and by investigating the asymptotic properties of the mass density, an inverse square law is seen to arise.

1. Introduction

The set of all possible metric tensors of general relativity make up the set $S_{4,1}$ of 4×4 real symmetric matrices of signature $(+++-)$. At a given instant of time any metric tensor $g_{\mu\nu}$ can be regarded as a continuous function of the space coordinates $\boldsymbol{x}$. Thus $g_{\mu\nu}$ represents a mapping from three-dimensional space R^3 into $S_{4,1}$. The set of all such mappings may be divided into classes, called homotopy classes, such that two mappings belong to the same homotopy class if and only if they can be continuously deformed into each other. It was pointed out by Finkelstein and Misner (1959, 1962) that this set of homotopy classes is in one–one correspondence with the group of integers. Thus a given metric can always be associated with an integer, which is called the 'kink number' of the metric. Provided that the space is assumed asymptotically flat§, so that $g_{\mu\nu}$ approaches a Lorentz metric at spatial infinity, then it follows that kink number is conserved as time evolves. It has been argued (Finkelstein and Misner 1962) that a kink may be reasonably interpreted as a fermion particle. For example, it has been shown (Williams 1971) that, for a one-kink metric, it is possible to introduce wavefunctions that are double valued under 2π rotation.

In this paper a model will be suggested for a one-kink metric, and the corresponding energy–momentum tensor will be examined.

2. The one-kink metric

$S_{4,1}$ is a fibre bundle with the three-dimensional rotation group SO_3 as base. Consider a degree one mapping from R^3 on to SO_3. If SO_3 is regarded as the base of $S_{4,1}$, then the mapping becomes a mapping from R^3 into $S_{4,1}$, and it can be shown (Steenrod 1951) to be a one-kink mapping. In what follows, we shall study the metric

$$g_{\mu\nu} = \delta_{\mu\nu} - 2\phi_\mu\phi_\nu. \tag{1}$$

† Present address: Chimie Physique 2, Université Libre de Bruxelles, 1050 Bruxelles, Belgium.

‡ Present address: Rutherford High Energy Laboratory, Chilton, Didcot, Berks, England.

§ The hypothesis of a closed universe would do equally well.

The symbol $\delta_{\mu\nu}$ is the Kronecker delta and the four real numbers $\{\phi_\mu\}$ represent an element of the three-sphere S^3 with $\phi_\mu\phi_\mu = 1$. (Greek indices will run over 1, 2, 3, 4, and italic indices over 1, 2, 3. The summation convention will be used throughout.) It is clear that $g_{\mu\nu}$ is a member of $S_{4,1}$ since it is possible to write

$$\|g_{\mu\nu}\| = P \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} P^{-1}$$

where

$$P = \begin{pmatrix} -\phi_4 & -\phi_3 & \phi_2 & -\phi_1 \\ \phi_3 & -\phi_4 & -\phi_1 & -\phi_2 \\ -\phi_2 & \phi_1 & -\phi_4 & -\phi_3 \\ \phi_1 & \phi_2 & \phi_3 & -\phi_4 \end{pmatrix}.$$

We shall choose the particular form of the $\{\phi_\mu\}$ to be

$$\phi_i = \frac{x^i}{r}\sin\alpha$$

$$\phi_4 = \cos\alpha \tag{2}$$

where $r = |\boldsymbol{x}|$, and α is any function of r such that

$$\alpha(0) = \pi \qquad \alpha(\infty) = 0. \tag{3}$$

The $\{\phi_\mu\}$ constitute a degree one mapping from R^3 into S^3. However, composition with the mapping of equation (1) really involves SO_3 instead of S^3, since giving a value to $g_{\mu\nu}$ only serves to determine the $\{\phi_\mu\}$ to within a $\pm$ sign. Thus the metric of equation (1), together with equation (2), is topologically equivalent to a degree one mapping from R^3 into the base of $S_{4,1}$. Thus $g_{\mu\nu}$ is a one-kink metric.

It is worth remarking that the orthogonal matrices P may be used to construct metrics corresponding to a higher number of kinks. For example, $g^N_{\mu\nu}$ given by

$$\|g^N_{\mu\nu}\| = P^N \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} P^{-N}$$

is an N kink metric.

3. Christoffel symbols and Ricci tensor

The Christoffel symbols for the metric $g_{\mu\nu}$ are readily calculated:

$$\Gamma^4_{44} = \sin^2(2\alpha)\alpha'$$

$$\Gamma^k_{44} = \Gamma^4_{4k} = -\tfrac{1}{2}\sin(4\alpha)\alpha'\left(\frac{x^k}{r}\right)$$

$$\Gamma^4_{ij} = (\cos^2 2\alpha + 1)\alpha'\left(\frac{x^i x^j}{r^2}\right) + \sin 2\alpha\left(\frac{\tau_{ij}}{r}\right)$$

$$\Gamma^k_{4j} = -\sin^2(2\alpha)\alpha'\left(\frac{x^j x^k}{r^2}\right)$$

$$\Gamma^k_{ij} = \tfrac{1}{2}\sin(4\alpha)\alpha'\left(\frac{x^i x^j x^k}{r^3}\right) + 2\sin^2\alpha\left(\frac{\tau_{ij} x^k}{r^2}\right)$$

where primes denote differentiation with respect to r and

$$\tau_{ij} = \delta_{ij} - \frac{x^i x^j}{r^2}.$$

Since $\det g = -1$, it follows that $g_{\mu\nu} = g^{\mu\nu}$ and $\Gamma^\mu_{\mu\nu} = 0$, all ν. The expression for the Ricci tensor simplifies to

$$R_{\mu\nu} = \partial_i \Gamma^i_{\mu\nu} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}$$

and it follows that

$$R_{44} = -\left(2\cos^2 2\alpha(\alpha')^2 - \sin 2\alpha\cos(2\alpha)\alpha'' + 2\sin 2\alpha\cos 2\alpha\frac{\alpha'}{r}\right)$$

$$R_{4j} = -\left(2\sin 2\alpha\cos 2\alpha(\alpha')^2 + \sin^2(2\alpha)\alpha'' + 2\sin^2 2\alpha\frac{\alpha'}{r}\right)\frac{x^j}{r}$$

$$R_{ij} = -\frac{R_{44}x^i x^j}{r^2} + \left(2\sin 2\alpha\frac{\alpha'}{r} + \frac{1-\cos 2\alpha}{r^2}\right)\tau_{ij}.$$

The curvature scalar becomes

$$R = 4\cos 2\alpha(\alpha')^2 + 2\sin(2\alpha)\alpha'' + 8\sin(2\alpha)\frac{\alpha'}{r} + \frac{2(1-\cos 2\alpha)}{r^2}$$

$$= \frac{1}{r^2}\partial_r^2\{r^2(1-\cos 2\alpha)\}.$$

4. The energy–momentum tensor

The energy–momentum tensor T^ν_μ is given by the Einstein equation $G^\nu_\mu = 8\pi G T^\nu_\mu$ where G is the gravitational constant and G^ν_μ is defined to be $R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R$. With the present sign convention T^4_4 is the negative of the mass density ρ. The components G^ν_μ are found to be

$$G^4_4 = -\frac{1}{r}\left(2\sin(2\alpha)\alpha' + \frac{1-\cos 2\alpha}{r}\right)$$

$$G^i_4 = G^4_i = 0$$

$$G^k_i = -\frac{1}{r}\left(2\sin(2\alpha)\alpha' + \frac{1-\cos 2\alpha}{r}\right)\frac{x^i x^k}{r^2} - \left(2\cos 2\alpha(\alpha')^2 + \sin(2\alpha)\alpha'' + 2\sin(2\alpha)\frac{\alpha'}{r}\right)\tau_{ik}.$$

From the G_4^4 above it follows that the mass density can be written

$$\rho = (4\pi G r^2)^{-1}\, \partial_r f(r)$$

where $f(r) = r\sin^2\alpha$. We wish to impose two conditions on ρ: (i) being the energy density of a single particle, ρ should be non-negative everywhere and (ii) its integral, representing the total mass, should be finite. The first condition implies that $f(r)$ must be a monotonic increasing function of r. From equation (3) we have $f(0) = 0$, so that the integral of ρ over all space is simply $G^{-1}f(\infty)$. By condition (ii) we set this to be a constant m, so that

$$f(\infty) = Gm. \tag{4}$$

This implies that

$$\sin^2\alpha \to \frac{Gm}{r} \qquad \text{as } r \to \infty.$$

This asymptotic condition has the consequence that the newtonian potential is reproduced. This can be seen by examining g_{44} through equation (1):

$$g_{44} = 1 - 2\phi_4^2 = -1 + 2\sin^2\alpha$$

so that $-\sin^2\alpha$ is just the required gravitational potential.

The size of the kink is associated, roughly speaking, with the value of r when α is $\pi/2$. The choice of this size lies with the choice of a functional form for $f(r)$. This large class of functions can be specified by four requirements: (i) monotonic increase, (ii) $f(\infty) =$ constant, (iii) $\max f(r) = 1$, and (iv) $f(r \to 0) = O(r)$. The last condition follows from equation (3). A simple example is

$$f(r) = \frac{2ar^2}{r^2 + a^2}.$$

In this particular example the 'radius' of the kink is just a. From equation (4) we see that this radius is related to the mass: $a = \frac{1}{2}Gm$. Since all parameters (besides a) introduced in the functional form of f are of the order of unity, it is not surprising that the size of the kink is of the order of the Schwarzschild radius.

5. Conclusions

This paper examines some properties of a metric describing a stationary kink at the origin. Although it is not clear to what extent this choice of metric typifies a one-kink system, it has the advantage of arising from the fibre bundle structure of $S_{4,1}$ in a rather natural and transparent manner. The energy–momentum tensor T_μ^ν is deduced and the form of T_4^4 examined. The requirements that the mass density should be positive everywhere and that the total mass be finite result in a restriction on the asymptotic form of the metric. This leads directly to Newton's inverse square force law.

Since no one will observe the gravitational field due to a single fermion, we must investigate the possibility of combining kink-fermions to form bodies of astronomical dimensions. The method mentioned in § 2 for forming a multikink metric is not sufficient if we wish to keep such simple properties as the newtonian force law.

Acknowledgments

One of us (JGW) would like to thank the Science Research Council for financial support in the form of a postdoctoral fellowship.

References

Finkelstein D and Misner C W 1959 *Ann. Phys., NY* **6** 230–43
—— 1962 *Les Theories Relativistes de la Gravitation* (Paris: Centre National de la Recherche Scientifique)
Steenrod N 1951 *Topology of Fibre Bundles* (Princeton: Princeton University Press)
Williams J G 1971 *J. math. Phys.* **12** 308–13